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K_{l3} form factor with $N_f = 2+1$ dynamical domain wall fermions: A progress report

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We present the latest results from the UKQCD/RBC collaborations for the K_{l3} form factor with $2+1$ flavours of dynamical domain wall quarks. Simulations are performed on $16^3 \times 32 \times 16$ and $24^3 \times 64 \times 16$ lattices with three values of the light quark mass, allowing for an extrapolation to the chiral limit. After interpolating to zero momentum transfer, we obtain the preliminary result $f_+^{K\pi}(0) = 0.9609(51)$ (or $\Delta f = -0.0161(51)$), which is in agreement with the result of Leutwyler & Roos.

I. INTRODUCTION

$K \to \pi l \nu$ (K_{l3}) decays provide an excellent avenue for an accurate determination of the Cabibbo-Kobayashi-Maskawa (CKM) [1] quark mixing matrix element, $|V_{us}|$. This is done by observing that the decay amplitude is proportional to $|V_{us}|^2 |f_+(q^2)|^2$, where $f_+(q^2)$ is the form factor defined from the $K \to \pi$ matrix element of the weak vector current, $V_\mu = \bar{s}\gamma_\mu u$

$$\langle \pi(p') \big| V_\mu \big| K(p) \rangle = (p_\mu + p'_\mu) f_+(q^2) + (p_\mu - p'_\mu) f_-(q^2) \,, \tag{1}$$

where $q^2 = (p - p')^2$.

In chiral perturbation theory (ChPT), $f_+(0)$ is expanded in terms of the light pseudoscalar meson masses, m_π, m_K, m_η

$$f_+(0) = 1 + f_2 + f_4 + \ldots, \quad (f_n = \mathcal{O}(m^n_{\pi,K,\eta})) \,. \tag{2}$$

Conservation of the vector current ensures that $f_+(0) = 1$ in the $SU(3)$ flavour limit. Additionally, as a result of the Ademollo-Gatto Theorem [2], which states that f_2 receives no contribution from local operators appearing in the effective theory, f_2 is determined unambiguously in terms of m_π, m_K and f_π, and takes the value $f_2 = -0.023$ at the physical values of the meson masses [3].

Our task is now reduced to one of finding

$$\Delta f = f_+(0) - (1 + f_2) \,. \tag{3}$$

In order to obtain a result for $f_+(0)$ which is accurate to $\sim$1%, it is sufficient to have a 20-30% error on Δf. Until recently, the standard estimate of $\Delta f = -0.016(8)$ was due to Leutwyler & Roos [3], however a more recent ChPT analysis favours a positive value, $\Delta f = 0.007(12)$ [4]. A calculation of Δf on the lattice is therefore essential.

The last few years have seen an improvement in the accuracy of lattice calculations of this quantity [5, 6, 7, 8, 9], with the results favouring a negative value for Δf in agreement with Leutwyler & Roos.

The UKQCD and RBC collaborations have embarked on a program to perform the first unitary (i.e. $N_f = 2+1$ flavour) lattice calculation of the $K \to \pi$ form factor using dynamical domain wall fermions at light quark masses and on large volumes. Preliminary results from this study were introduced in Ref. [10]. The work presented here represents an improvement on the earlier analysis through a more in depth analysis of the systematic errors involved in the extraction of $f_+(0)$ from lattice correlation functions.

II. LATTICE TECHNIQUES

A. Parameters

We simulate with $N_f = 2+1$ dynamical flavours generated with the Iwasaki gauge action [11] at $\beta = 2.13$, which corresponds to an inverse lattice spacing $a^{-1} \approx 1.62(4)$ GeV [12], and the domain wall fermion action [13] with domain wall height $M_5 = 1.8$ and fifth dimension length $L_s = 16$. This results in a residual mass of $am_{\rm res} \approx 0.00308(4)$ [12]. The simulated strange quark mass, $am_s = 0.04$, is very close to its physical value [12], and we choose three values for the light quark masses, $am_{ud} = 0.03, 0.02, 0.01$, which correspond to pion masses $m_\pi \approx 630$, 520, 390 MeV [12]. The calculations are performed on two volumes, $16^3 \times 32$ and $24^3 \times 64$, at each quark mass. See [12] for further simulation details.

B. Extracting form factors

The techniques we use to calculate $f_+(0)$ are similar to those originally proposed in [5] and have been outlined in detail in Ref. [9]. We restrict ourselves here to highlighting the main points.

*Talk presented by J. Zanotti at CKM2006, Nagoya, Japan

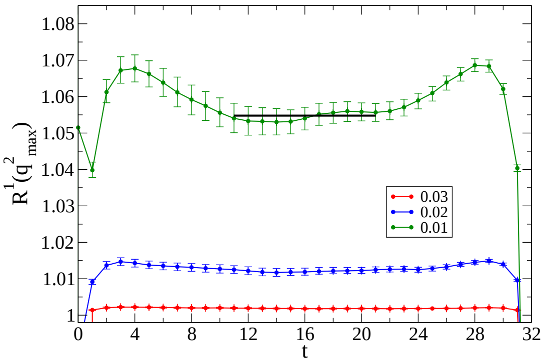

FIG. 1: Ratio for $f_0(q^2_{\rm max})$, $R(t',t)$, as defined in Eq. (5), for three simulated light masses $am_{ud} = 0.03, 0.02, 0.01$ on a $24^3 \times 64$ volume. Further simulation parameters can be found in [12].

We start by rewriting the vector form factors given in Eq. (1) to define the scalar form factor

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2) \ , \qquad (4)$$

which can be obtained on the lattice at $q^2_{\rm max} = (m_K - m_\pi)^2$ with high precision from the following ratio [14]

$$\begin{aligned} R(t',t) \quad &= \quad \frac{C_4^{K\pi}(t',t;\vec{0},\vec{0}) C_4^{\pi K}(t',t;\vec{0},\vec{0})}{C_4^{KK}(t',t;\vec{0},\vec{0}) C_4^{\pi\pi}(t',t;\vec{0},\vec{0})} \\ &\xrightarrow[t,(t'-t)\to\infty]{} \frac{(m_K + m_\pi)^2}{4 m_K m_\pi} \left| f_0(q^2_{\rm max}) \right|^2 \ , \qquad (5) \end{aligned}$$

where the three-point function is defined as

$$\begin{aligned} C_\mu^{PQ}(t',t,\vec{p}',\vec{p}) = \sum_{\vec{x},\vec{y}} e^{-i\vec{p}'(\vec{y}-\vec{x})} e^{-i\vec{p}\vec{x}} \times \qquad (6) \\ \langle 0 | \mathcal{O}_Q(t',\vec{y}) | Q \rangle \langle Q | V_\mu(t,\vec{x}) | P \rangle \langle P | \mathcal{O}_P^\dagger(0) | 0 \rangle \ , \end{aligned}$$

with $P, Q = \pi$ or K and $\mathcal{O}_{\pi(K)}$ is a local interpolating operator for a pion(kaon). We note that $R(t',t) = 1$ in the $SU(3)_{\rm flavour}$ symmetric limit, hence any deviations from unity are purely due to $SU(3)_{\rm flavour}$ symmetry breaking effects.

In Fig. 1 we display our results for $R(t',t)$ for each of the simulated quark masses as obtained on the $24^3 \times 64$ lattices.

It is immediately obvious that $f_0(q^2_{\rm max})$ can be measured with a very high level of statistical accuracy. We also note that the ratio becomes larger the further we move away from the $SU(3)_{\rm flavour}$ limit. Since there is no spatial momentum involved in this ratio, the results obtained on the two different volumes should agree, and any difference can only be due to finite size effects. We find that within statistical errors, finite size effects on $f_0(q^2_{\rm max})$ are negligible [10].

In order to to extract the form factor at $q^2 = 0$, we need to obtain results at finite $q^2 < 0$ and by studying the q^2 dependence we are then in a position to interpolate to $q^2 = 0$. We calculate the form factor $f_0(q^2 < 0)$ using the ratio method outlined in Ref. [5, 9] which has the advantage that no knowledge of Z_V is required. We simulate with two unique choices of the three-momentum transfer, $\vec{q} = 2\pi/L(1,0,0)$ and $\vec{q} = 2\pi/L(1,1,0)$, where L is the spatial extent of our lattice. We take advantage of rotational symmetry to improve the signal,i.e. we use all 6 and 12 permutations of (1,0,0) and (1,1,0), respectively. The ratio method also allows $f_0(q^2)$ to be extracted from the $\pi \to K$ matrix element which, due to the difference between the π and K masses, doubles the number of q^2 values. Finally, since (spatial) momentum on the lattice is proportional to the inverse (spatial) lattice size, our simulations on two different volumes provides access to additional q^2.

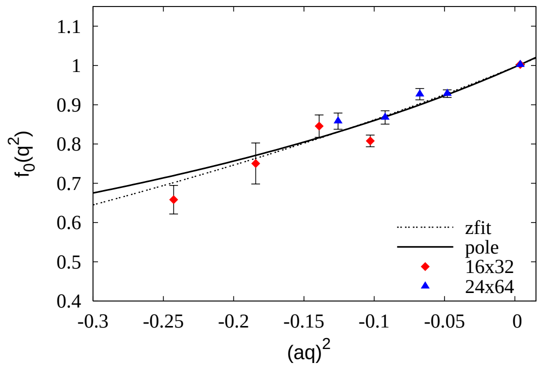

FIG. 2: Scalar form factor $f_0(q^2)$ for bare quark mass $am_{ud} = 0.02$. Results are obtained on two volumes $V = 16^3 \times 32$ (red diamonds) and $V = 24^3 \times 64$ (blue triangles). The solid line is the result of a fit using a monopole ansatz (Eq. (7)) while the dotted line is the result of a fit using a zfit ansatz (Eq. (12)).

III. RESULTS

A. Interpolation to $q^2 = 0$

We present our results obtained on each volume for $f_0(q^2)$ in Fig. 2 for quark mass $am_{ud} = 0.02$. In the intermediate q^2-range we see good agreement between the results obtained on the two different volumes, indicating that finite size effects are negligible, at least for the quark masses considered here. This means that we now have results over a large range of q^2 to fit to.

It is not immediately clear what is the best method for fitting $f_0(q^2)$, however previous work [5, 9] has found that lattice results are described very well by a monopole form

$$f_0(q^2) = \frac{f_0(0)}{(1 - q^2/M^2)} \ , \qquad (7)$$

where M is the monopole mass. From Fig. 2 we see that our data is also described well by Eq. 7 (solid line). From this fit, we are able to calculate $f_0(0)$ for our three simulated quark masses.

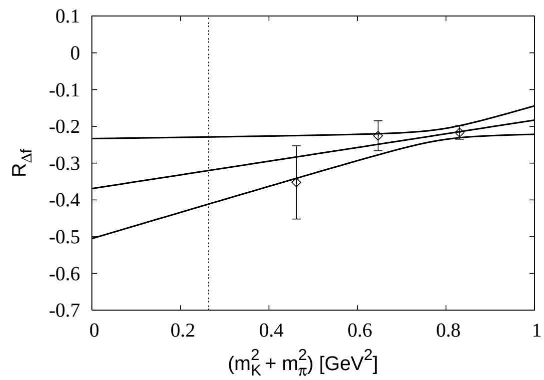

FIG. 3: Chiral extrapolation of Δf using Eq. (9).

B. Chiral Extrapolation

Now that we have obtained results for $f_+(0) = f_0(0)$ at three different quark masses, we are in a position to attempt an extrapolation to the physical pion mass. Inserting these results into the expression given in Eq. (3), together with f_2 calculated at the simulated quark masses using the ChPT formula [3, 15], we are now left with the task of chirally extrapolating Δf.

The Ademollo-Gatto Theorem implies that to leading order $\Delta f \propto (m_s - m_{ud})^2$, hence we attempt a chiral extrapolation using

$$\Delta f = a + B(m_s - m_{ud})^2 \; . \tag{8}$$

Note that in the $SU(3)_{\text{flavour}}$ limit, $\Delta f = 0$, so we expect that a fit to our data should produce $a \approx 0$, and indeed we find this to be the case. In the chiral limit we find $\Delta f = -0.0146(28)$.

In order to attempt to take into account higher terms in the chiral expansion of Δf, it has been noted that it is convenient to consider an extrapolation of the ratio [5, 9]

$$R_{\Delta f} = \frac{\Delta f}{(m_K^2 - m_\pi^2)^2} = a + b(m_K^2 + m_\pi^2) \; . \tag{9}$$

We show the extrapolation using this form in Fig 3 from which we extract a result at the physical meson masses (vertical dotted line), $\Delta f = -0.0161(46)$.

C. Estimating Systematic Errors

As we have seen in the previous section, with only three quark masses it is difficult to distinguish between different chiral extrapolation ansätze. Hence, we quote the result obtained using Eq. (9), and use the difference between the results obtained from the two extrapolations (0.0015) as an estimate of the systematic error due to the chiral extrapolation.

Another possible source of systematic error comes from our choice of a monopole fit for the q^2 dependence of $f_0(q^2)$. In order to estimate the systematic error due to this choice, we fit $f_0(q^2)$ at each quark mass with a linear form

$$f_0(q^2) = f_0(0) + a_1 q^2 \; , \tag{10}$$

a quadratic form

$$f_0(q^2) = f_0(0) + a_1 q^2 + a_2 q^4 \; , \tag{11}$$

and a parameterisation proposed in Ref. [16]

$$f_0(t) = \frac{1}{\phi(t, t_0, Q^2)} \sum_{k=0}^{\infty} a_k(t_0, Q^2) z(t, t_0)^k \; , \tag{12}$$

where $|z| < 1$ is a mapping of $t = q^2$ onto the unit circle in the complex plane

$$t \to z(t, t_0) \equiv \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}} \; ,$$

and $t_\pm \equiv (m_K \pm m_\pi)^2$. We use $t_0 = t_+(1 - \sqrt{1 - t_-/t_+})$ as the point that maps onto $z = 0$, and

$$\begin{aligned} \phi(t, t_0, Q^2) \;&=\; \sqrt{\frac{3 t_+ t_-}{32\pi}} \, \frac{z(t,0)}{-t} \, \frac{z(t, -Q^2)}{-Q^2 - t} \left(\frac{z(t, t_0)}{t_0 - t} \right)^{-1/2} \\ &\times \left(\frac{z(t, t_-)}{t_- - t} \right)^{-1/4} \frac{\sqrt{t_+ - t}}{(t_+ - t_0)^{1/4}} \; , \end{aligned}$$

with $Q^2 = 2/a^2$ and $a^{-1} = 1.62$ GeV is the lattice spacing. We note that employing different values for t_0 and Q^2 simply affects the fitted values of the coefficients, a_k, and not the result for $f_0(0)$.

As an example, the dotted line in Fig. 2 shows the result of a fit to $f_0(q^2)$ for quark mass $am_{ud} = 0.02$ using the parameterisation in Eq. (12) truncated to include terms up to $k = 2$.

We find that all four parameterisations describe our data reasonably well, except the linear ansatz which leads to a large χ^2/dof on the $am = 0.01$ dataset. We take the difference between the result from the pole fit and the z fit as the estimate of the systematic error due to the q^2 dependence.

In Fig. 4, we provide a comparison of the value of Δf at the physical meson masses as obtained from all four parameterisations. The blue triangles indicate the results using Eq. (8) for the chiral extrapolation, while the red diamonds use Eq. (9). The four sets of points are, moving from left to right, linear (10), quadratic (11), z (12) and pole (7) fits, respectively. We observe that the systematic error due to the q^2 dependence is comparable to that from the chiral extrapolation, and both are small when compared to the statistical error. Hence, one might hope that these systematic errors can be reduced along with the statistical error by improving the statistics.

Finally, since we only have results at one lattice spacing, we are unable to extrapolate to the continuum limit. However, lattice artefacts are formally of $O(a^2\Lambda^2_{QCD})$; assuming $\Lambda^2_{QCD} \sim 300$ MeV we, *tentatively*, estimate these

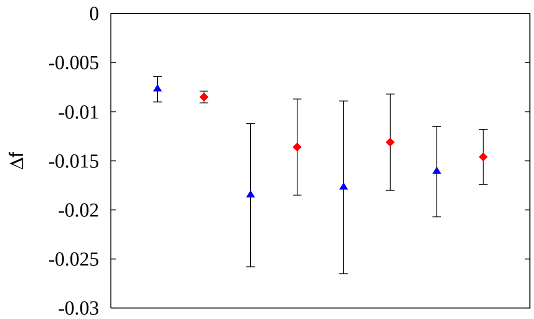

FIG. 4: Comparison of Δf at the physical meson masses from all four parameterisations. Blue triangles indicate the results using Eq. (8) for the chiral extrapolation, while the red diamonds use Eq. (9). The four sets of points are, moving from left to right, linear, quadratic, z and pole fits, respectively.

at $\approx 4\%$. Hence our preliminary result is

$$\Delta f = -0.0161(46)(15)(16)(7) \Rightarrow f_+^{K\pi}(0) = 0.9609(51)\ , \tag{13}$$

where the first error is statistical, and the second, third and fourth are estimates of the systematic errors due to the chiral extrapolation, q^2 dependence and lattice artefacts, respectively. This result agrees very well with the old value of Leutwyler & Roos [3] and earlier lattice calculations [5, 6, 7, 8, 9].

While this result for $f_+(0)$ is smaller than our earlier preliminary analysis (0.9680(16)) [10] and has a larger error, we believe that through a more rigorous investigation of the systematics involved in extracting this result, we are now much closer to having a finalised result which we plan to publish soon.

Using $|V_{us} f_+(0)| = 0.2169(9)$ from the experimental decay amplitude [17]:

$$|V_{us}| = 0.2257(9)_{\rm exp}(12)_{f_+(0)} \tag{14}$$

we find $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - \delta, \quad \delta = 0.00076(62)$, compared with the PDG(2006) result, $\delta = 0.0008(10)$.

IV. SUMMARY AND FUTURE WORK

We have presented a preliminary result for $\Delta f = f_+(0) - (1 + f_2)$ using $N_f = 2+1$ dynamical domain wall fermions with three choices for the light quark masses. Our result $\Delta f = -0.0161(46)(15)(16)(7)$ agrees very well with the Leutwyler & Roos result [3] and confirms the trend of other lattice results [5, 6, 7, 8, 9] which prefer a negative value for Δf. We performed our simulations with matched parameters on two volumes and we observe no obvious finite size effects.

We are in the process of improving this result by looking at ways of decreasing the error on the point at $am_{ud} = 0.01$. We will also soon have a result at a lighter quark mass ($am_{ud} = 0.005$) which will assist in improving the chiral extrapolation. Additionally, this result has been obtained at a single value of the lattice spacing, so future simulations will need to be performed at least at one more lattice spacing to investigate scaling behaviour.

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